

NORMAL EXTENSIONS

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Abstract. Let L_0 be a densely defined minimal linear operator in a Hilbert space H . We prove theorem that if there exists at least one correct extension L_S of L_0 with the property $D(L_S) = D(L_S^*)$, then we can describe all correct extensions L with the property $D(L) = D(L^*)$. We also prove that if L_0 is formally normal and there exists at least one correct normal extension L_N , then we can describe all correct normal extensions L of L_0 . As an example, the Cauchy-Riemann operator is given.

1 Introduction

Let us present some definitions, notation, and terminology.

In a Hilbert space H , we consider a linear operator L with domain $D(L)$ and range $R(L)$. By the *kernel* of the operator L we mean the set

$$\text{Ker } L = \{f \in D(L) : Lf = 0\}.$$

Definition 1. An operator L is called a *restriction* of an operator L_1 , and L_1 is called an *extension* of an operator L , briefly $L \subset L_1$, if:

- 1) $D(L) \subset D(L_1)$,
- 2) $Lf = L_1f$ for all f from $D(L)$.

Definition 2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if $\overline{R(L_0)} \neq H$ and there exists a bounded inverse operator L_0^{-1} on $R(L_0)$.

Definition 3. A linear closed operator \hat{L} in a Hilbert space H is called *maximal* if $R(\hat{L}) = H$ and $\text{Ker } \hat{L} \neq \{0\}$.

Definition 4. A linear closed operator L in a Hilbert space H is called *correct* if there exists a bounded inverse operator L^{-1} defined on all of H .

Definition 5. We say that a correct operator L in a Hilbert space H is a *correct extension* of minimal operator L_0 (*correct restriction* of maximal operator \hat{L}) if $L_0 \subset L$ ($L \subset \hat{L}$).

Definition 6. We say that a correct operator L in a Hilbert space H is a *boundary correct* extension of a minimal operator L_0 with respect to a maximal operator \hat{L} if L is simultaneously a correct restriction of the maximal operator \hat{L} and a correct extension of the minimal operator L_0 , that is, $L_0 \subset L \subset \hat{L}$.

At the beginning of the 1950s, Vishik [10] extended the theory of self-adjoint extensions of von Neumann–Krein symmetric operators to nonsymmetric operators in Hilbert space.

At the beginning of the 1980s, M. Otelbaev and his disciples proved abstract theorems that allows us to describe all correct extensions of some minimal operator using any single known correct extension in terms of an inverse operator. Here such extensions need not be restrictions of a maximal operator. Similarly, all possible correct restrictions of some maximal operator that need not be extensions of a minimal operator were described (see [7]). For convenience, we present the conclusions of these theorems.

Let \widehat{L} be a maximal linear operator in a Hilbert space H , let L be any known correct restriction of \widehat{L} , and let K be an arbitrary linear bounded (in H) operator satisfying the following condition:

$$R(K) \subset \text{Ker } \widehat{L}. \quad (1.1)$$

Then the operator L_K^{-1} defined by the formula

$$L_K^{-1}f = L^{-1}f + Kf, \quad (1.2)$$

describes the inverse operators to all possible correct restrictions L_K of \widehat{L} , i.e., $L_K \subset \widehat{L}$.

Let L_0 be a minimal operator in a Hilbert space H , let L be any known correct extension of L_0 , and let K be a linear bounded operator in H satisfying the conditions

- a) $R(L_0) \subset \text{Ker } K$,
- b) $\text{Ker } (L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators to all possible correct extensions L_K of L_0 .

Let L be any known boundary correct extension of L_0 , i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [10]. Let K be a linear bounded (in H) operator satisfying the conditions

- a) $R(L_0) \subset \text{Ker } K$,
- b) $R(K) \subset \text{Ker } \widehat{L}$,

then the operator L_K^{-1} defined by formula (1.2) describes the inverse operators to all possible boundary correct extensions L_K of L_0 .

Self-adjoint and unitary operators are particular cases of normal operators. A bounded linear operator N in a Hilbert space H is called *normal* if it commutes with its adjoint:

$$N^*N = NN^*.$$

The theory of bounded normal operators are sufficiently developed.

Consider an unbounded linear operator A in a Hilbert space H .

Definition 7. A densely defined closed linear operator A in a Hilbert space H is called *formally normal* if

$$D(A) \subset D(A^*), \quad \|Af\| = \|A^*f\| \quad \text{for all } f \in D(A).$$

Definition 8. A formally normal operator A is called *normal* if

$$D(A) = D(A^*).$$

Normal extensions of formally normal operators have been studied by many authors (see [1], [5], [6], [9]). Questions the existence of a normal extension and the description of the domains of normal extensions of a formally normal operator were considered.

The spectral properties of the correct restrictions and extensions were systematically studied by the author (see [2]–[4]). In these works a class of operators K that provides Volterra, the completeness of root vectors, and the dissipativity of the correct restrictions and extensions were described. The present paper is devoted to the description of correct normal extensions in terms of the operator K .

2 Coincidence criterion of $D(L)$ with $D(L^*)$

We consider a densely defined minimal linear operator L_0 in a Hilbert space H . Let M_0 be a minimal operator with $D(M_0) = D(L_0)$ that is connected with L_0 by the relation $(L_0u, v) = (u, M_0v)$ for all u, v from $D(L_0)$. Then the maximal operator $\widehat{L} = M_0^*$ is an extension of L_0 , and the maximal operator $\widehat{M} = L_0^*$ is an extension of M_0 . The following statement is true.

Assertion 1. *If there exists a correct extension L_S of the minimal operator L_0 with the property $D(L_S) = D(L_S^*)$, then the operator L_S is the boundary correct extension, i.e., $L_0 \subset L_S \subset \widehat{L}$.*

Proof. From $L_0 \subset L_S$ it follows that $L_S^* \subset L_0^* = \widehat{M}$. From $D(L_S) = D(L_S^*)$ and the fact that $D(M_0) \subset D(L_S^*)$ we have

$$M_0 \subset L_S^* \subset \widehat{M}.$$

Then $L_0 \subset L_S \subset \widehat{L}$. The assertion is proved. \square

Let there be one fixed correct extension L_S of L_0 such that $D(L_S) = D(L_S^*)$. Then we can describe the inverses to all boundary correct extensions L in the following form

$$u = L^{-1}f = L_S^{-1}f + Kf \quad \text{for all } f \in H, \quad (2.1)$$

where K is an arbitrary bounded operator in a Hilbert space H that

$$R(K) \subset \text{Ker } \widehat{L} \quad \text{and} \quad R(L_0) \subset \text{Ker } K.$$

Each such operator K defines one boundary correct extension and there do not exist other boundary correct extensions.

Let us equip $D(\widehat{L})$ with the graph norm $\|u\|_G = (\|u\|^2 + \|\widehat{L}u\|^2)^{1/2}$. Since \widehat{L} is a closed operator, we obtain a Hilbert space with the scalar product

$$(u, v)_G = (u, v) + (\widehat{L}u, \widehat{L}v) \quad \text{for all } u, v \text{ from } D(\widehat{L}).$$

Let us denote this space by $G_{\widehat{L}}$. The domain $D(L_S)$ of the correct restriction L_S is a subspace in $G_{\widehat{L}}$. Therefore, there exists a projection operator of $G_{\widehat{L}}$ on the subspace $D(L_S)$. As such a projection operator, we take $L_S^{-1}\widehat{L}$. Then the projection $\Gamma_{L_S} = I - L_S^{-1}\widehat{L}$ of $G_{\widehat{L}}$ on the subspace $\text{Ker } \widehat{L}$. It is obvious that

$$\text{Ker } \Gamma_{L_S} = D(L_S) \quad \text{and} \quad R(\Gamma_{L_S}) = \text{Ker } \widehat{L}.$$

All boundary correct extensions (2.1) transforms into

$$L^{-1}f = L_S^{-1}f + Kf = L_S^{-1}f + K\widehat{L}L_S^{-1}f = (I + K\widehat{L})L_S^{-1}f \quad \text{for all } f \text{ from } H,$$

where I is the identity operator in H . In virtue of $D(L) \subset D(\widehat{L})$, we have

$$\widehat{L}u = f \quad \text{for all } f \text{ from } H, \quad u \text{ from } D(L)$$

where

$$D(L) = \{u \in D(\widehat{L}) : (I - K\widehat{L})u \in D(L_S)\}.$$

It is easy to see that the operator K defines the domain of L , as (see [3])

$$(I - K\widehat{L})D(L) = D(L_S), \quad (I + K\widehat{L})D(L_S) = D(L), \quad (I - K\widehat{L}) = (I + K\widehat{L})^{-1}.$$

Therefore, all boundary correct extensions L are differed from fixed boundary correct extension L_S only the domain. The bounded (in $G_{\widehat{L}}$) operator $I - K\widehat{L}$ maps $D(L)$ onto $D(L_S)$ in a one-to-one fashion. Then the domain of L can be defined as follows:

$$D(L) = \{u \in D(\widehat{L}) : \Gamma_{L_S}(I - K\widehat{L})u = 0\}.$$

There exists one more representation of the domain of L

$$D(L) = \{u \in D(\widehat{L}) : ((I - K\widehat{L})u, L_S^*v) = (\widehat{L}u, v) \text{ for all } v \text{ from } D(L_S^*) \}.$$

Similarly we can define

$$\Gamma_{L_S^*} = I - L_S^{*-1}\widehat{M}$$

and

$$D(L^*) = \{u \in D(\widehat{M}) : \Gamma_{L_S^*}(I - K^*\widehat{M})u = 0\}.$$

Now we can formulate the following result:

Theorem 2. *Let there be a correct extension L_S of the minimal operator L_0 with $D(L_S) = D(L_S^*)$, then any other correct extension L has the property $D(L) = D(L^*)$ if and only if $L_0 \subset L \subset \widehat{L}$ and the operator K from the formula (2.1) satisfies the conditions*

$$R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M}),$$

and

$$\begin{cases} \Gamma_{L_S}(I - K\widehat{L})u = 0, \\ \Gamma_{L_S}K^*\widehat{M}u = K\widehat{L}u, \quad \text{for all } u \in D(\widehat{L}) \cap D(\widehat{M}), \end{cases} \quad (2.2)$$

where $\Gamma_{L_S} = I - L_S^{-1}\widehat{L}$ is the projection defined above.

Proof. Let $D(L) = D(L^*)$. In view of Assertion 1, the operators L_S and L turn out to be boundary correct extensions of L_0 , i.e., $L_0 \subset L_S \subset \widehat{L}$ and $L_0 \subset L \subset \widehat{L}$. The inverse to the arbitrary boundary correct extension L has the form (2.1). Then

$$(L^*)^{-1}g = (L_S^*)^{-1}g + K^*g \quad \text{for all } g \in H.$$

The condition $D(L) = D(L^*)$ is equivalent to

$$L_S^{-1}f + Kf = (L_S^*)^{-1}g + K^*g, \quad (2.3)$$

where for each $f \in H$ there exists $g \in H$ and vice versa, for each $g \in H$ there exists $f \in H$ that the equality (2.3) is fulfilled. It follows from (2.3) that

$$R(K^*) \subset D(\widehat{L}) \quad \text{and} \quad R(K) \subset D(\widehat{M}).$$

Then we get

$$R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M}).$$

Acting on both sides of equality (2.3) by the operator \widehat{L} , we obtain

$$f = L_S(L_S^*)^{-1}g + \widehat{L}K^*g, \quad \text{for all } g \in H.$$

Substituting f into (2.3), we obtain the equality

$$L_S^{-1}\widehat{L}K^*g + KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g = K^*g.$$

It follows that

$$(I - L_S^{-1}\widehat{L})K^*g = K\widehat{L}((L_S^*)^{-1} + K^*)g.$$

This means that

$$(I - L_S^{-1}\widehat{L})K^*g = K\widehat{L}(L^*)^{-1}g.$$

If $L^{*-1}g$ is replaced by u , then

$$(I - L_S^{-1}\widehat{L})K^*\widehat{M}u = K\widehat{L}u, \quad u \in D(L^*).$$

Since $D(L) = D(L^*)$ we obtain $\Gamma_{L_S}K^*\widehat{M}u = K\widehat{L}u$ for all u from $D(L)$. This is equivalent to the condition (2.2).

We now prove a converse of this theorem. Let $L_0 \subset L \subset \widehat{L}$ and the operator K from the formula (2.1) satisfies the conditions $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$, and (2.2). Hence, it is easy to see that

$$D(L) \cup D(L^*) \subset D(\widehat{L}) \cap D(\widehat{M}).$$

Since $Lu = f$ for all $u \in D(L)$, we may replace Lu by f in the second equation of the condition (2.2). Then

$$\Gamma_{L_S}K^*\widehat{M}L^{-1}f = Kf \quad \text{for all } f \in H.$$

Acting on both sides of this equality by the projection $\Gamma_{L_S^*}$, we obtain

$$K^*\widehat{M}L^{-1}f = (I - (L_S^*)^{-1}\widehat{M})Kf \quad \text{for all } f \in H.$$

Note that

$$K^*L_S^*L_S^{-1}f + K^*\widehat{M}Kf + (L_S^*)^{-1}\widehat{M}Kf = Kf.$$

Adding the bounded operator $L_S^{-1}f$ to both sides, we get

$$(L_S^*)^{-1}L_S^*L_S^{-1}f + K^*L_S^*L_S^{-1}f + K^*\widehat{M}Kf + (L_S^*)^{-1}\widehat{M}Kf = Kf + L_S^{-1}f.$$

It follows that

$$(L_S^*)^{-1}(L_S^*L_S^{-1} + \widehat{M}K)f + K^*(L_S^*L_S^{-1} + \widehat{M}K)f = L^{-1}f \quad \text{for all } f \in H.$$

If we denote by

$$g = L_S^*L_S^{-1}f + \widehat{M}Kf \quad \text{for all } f \in H,$$

then we have

$$(L^*)^{-1}g = L^{-1}f \quad \text{for all } f \in H.$$

It follows that $D(L) \subset D(L^*)$. Acting on both sides of the equations (2.2) by the projection $\Gamma_{L_S^*}$, we get

$$\begin{cases} \Gamma_{L_S^*}(I - K\widehat{L})u = 0, \\ \Gamma_{L_S^*}K\widehat{L}u = K^*\widehat{M}u \quad \text{for all } u \in D(\widehat{L}) \cap D(\widehat{M}). \end{cases}$$

By the second equation of the given system, we can rewrite this system of equations in the form

$$\begin{cases} \Gamma_{L_S^*}(I - K^*\widehat{M})u = 0, \\ \Gamma_{L_S^*}K\widehat{L}u = K^*\widehat{M}u \quad \text{for all } u \in D(\widehat{L}) \cap D(\widehat{M}). \end{cases}$$

The first equation of this system means that u belongs to $D(L^*)$. Then we denote $L^*u = g$. Therefore, $u = (L^*)^{-1}g$ for all g from H . Then the second equation of this system has the form

$$\Gamma_{L_S^*}K\widehat{L}(L^*)^{-1}g = K^*\widehat{M}(L^*)^{-1}g \quad \text{for all } g \in H.$$

Acting on both sides of this equality by the projection Γ_{L_S} , we obtain

$$K\widehat{L}(L^*)^{-1}g = (I - (L_S)^{-1}\widehat{L})K^*g \quad \text{for all } g \in H.$$

Note that

$$KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g + L_S^{-1}\widehat{L}K^*g = K^*g.$$

Adding the bounded operator $(L_S^*)^{-1}g$ to both sides, we get

$$L_S^{-1}L_S(L_S^*)^{-1}g + KL_S(L_S^*)^{-1}g + K\widehat{L}K^*g + L_S^{-1}\widehat{L}K^*g = K^*g + (L_S^*)^{-1}g.$$

It follows that

$$L_S^{-1}(L_S(L_S^*)^{-1} + \widehat{L}K^*)g + K(L_S(L_S^*)^{-1} + \widehat{L}K^*)g = (L^*)^{-1}g \quad \text{for all } g \in H.$$

If we denote by

$$f = (L_S(L_S^*)^{-1} + \widehat{L}K^*)g \quad \text{for all } g \in H,$$

then we have

$$L^{-1}f = (L^*)^{-1}g \quad \text{for all } g \in H.$$

It follows that $D(L^*) \subset D(L)$. The theorem is proved. \square

3 Normality criterion of correct extensions

Let L_0 be a formally normal minimal operator in a Hilbert space H . An operator M_0 is the restriction of $L_0^* = \widehat{M}$ to $D(L_0)$. Then $\widehat{L} = M_0^*$ defines the maximal operator that $L_0 \subset \widehat{L}$. Let there be at least one normal correct extension L_N of the formally normal minimal operator L_0 . In view of Assertion 1, we have that $L_0 \subset L_N \subset \widehat{L}$, i.e., L_N is the boundary correct extension. Then the inverses to all boundary correct extensions L of L_0 have the form

$$u = L^{-1}f = L_N^{-1}f + Kf \quad \text{for all } f \in H, \quad (3.1)$$

where K is an arbitrary bounded operator in a Hilbert space H that $R(K) \subset \text{Ker } \widehat{L}$ and $R(L_0) \subset \text{Ker } K$. Then the direct operator L acts as

$$\widehat{L}u = f \quad \text{for all } f \in H,$$

on the domain

$$D(L) = \{u \in D(\widehat{L}) : \Gamma_{L_N}(I - K\widehat{L})u = 0\},$$

where the projection $\Gamma_{L_N} = I - L_N^{-1}\widehat{L}$ is the bounded operator in the space $G_{\widehat{L}}$. It is known that

$$\text{Ker } \Gamma_{L_N} = D(L_N) \quad \text{and} \quad R(\Gamma_{L_N}) = \text{Ker } \widehat{L}.$$

Theorem 3. *Let there be one correct normal extension L_N of the formally normal minimal operator L_0 in a Hilbert space H . Then any other correct extension L of L_0 is normal if and only if $L_0 \subset L \subset \widehat{L}$ and operator K from the formula (3.1) satisfies the conditions:*

$$\begin{cases} \Gamma_{L_N}(I - K\widehat{L})u = 0, \\ \Gamma_{L_N}K^*\widehat{M}u = K\widehat{L}u \quad \text{for all } u \in D(\widehat{L}) \cap D(\widehat{M}), \end{cases} \quad (3.2)$$

and

$$\widehat{L}K^* = (\widehat{M}K)^*, \quad (3.3)$$

where $\Gamma_{L_N} = I - L_N^{-1}\widehat{L}$ is projection on $\text{Ker } \widehat{L}$.

Proof. Let L be a normal correct extension of the formally normal operator L_0 . In view of Theorem 2, the conditions $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and (3.2) will be fulfilled. The normality of L^{-1} follows from the normality of L :

$$L^{-1}(L^*)^{-1} = (L^*)^{-1}L^{-1}.$$

By virtue of (3.1), we obtain

$$(L_N^{-1} + K)((L_N^*)^{-1} + K^*)f = ((L_N^*)^{-1} + K^*)(L_N^{-1} + K)f \quad \text{for all } f \in H.$$

It follows that

$$L_N^{-1}K^*f + K(L_N^*)^{-1} + KK^*f = (L_N^*)^{-1}Kf + K^*L_N^{-1}f + K^*Kf. \quad (3.4)$$

Acting on both sides of the equality (3.4) by the operator \widehat{L} , we get

$$K^*f = L_N(L_N^*)^{-1}Kf + \widehat{L}K^*L_N^{-1}f + \widehat{L}K^*Kf.$$

Taking conjugates of both sides of the equality above, we have

$$Kf = K^*(L_N(L_N^*)^{-1})^*f + (L_N^*)^{-1}(\widehat{L}K^*)^*f + K^*(\widehat{L}K^*)^*f \quad \text{for all } f \in H.$$

Acting on both sides by the operator \widehat{M} , we obtain

$$\widehat{M}Kf = (\widehat{L}K^*)^*f \quad \text{for all } f \in H.$$

This is equivalent to

$$\widehat{L}K^* = (\widehat{M}K)^*.$$

Let us prove the converse. Suppose that the conditions of Theorem 3 hold. From the conditions $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and (3.2), in view of Theorem 2, we have that $D(L) = D(L^*)$. Then for all $f \in H$ there exists $g \in H$ such that $L^{-1}f = (L^*)^{-1}g$. It can be rewritten in the form

$$L_N^{-1}f + Kf = (L_N^*)^{-1}g + K^*g. \quad (3.5)$$

Acting on both sides by the operator \widehat{M} , we get

$$g = L_N^*L_N^{-1}f + \widehat{M}Kf.$$

Substituting g into (3.5), we have

$$Kf = (L_N^*)^{-1}\widehat{M}Kf + K^*L_N^*L_N^{-1}f + K^*\widehat{M}Kf \quad \text{for all } f \in H.$$

Then

$$K^*f = (\widehat{M}K)^*L_N^{-1}f + (L_N^*L_N^{-1})^*Kf + (\widehat{M}K)^*Kf \quad \text{for all } f \in H. \quad (3.6)$$

Let us show that

$$(L_N^*L_N^{-1})^* = L_N(L_N^*)^{-1}.$$

It is known that if A is a closed operator, B is bounded in H and AB is densely defined in H , then

$$(AB)^* = \overline{B^*A^*},$$

where the overbar denotes the closure operator. Note that

$$L_N^*L_N^{-1} \supset L_N^{-1}L_N^*.$$

Then

$$(L_N^* L_N^{-1})^* = \overline{(L_N^*)^{-1} L_N} \subset L_N (L_N^*)^{-1}.$$

Taking into account the fact that $L_N (L_N^*)^{-1}$ is the bounded operator that coincides with $(L_N^*)^{-1} L_N$ on the dense set $D(L_N)$, then we obtain that

$$L_N (L_N^*)^{-1} = \overline{(L_N^*)^{-1} L_N} = (L_N^* L_N^{-1})^*.$$

Then, taking into account (3.3), the equality (3.6) can be rewritten in the form

$$K^* f = \widehat{L} K^* L_N^{-1} f + L_N (L_N^*)^{-1} K f + \widehat{L} K^* K f \quad \text{for all } f \in H.$$

Adding $(L_N^*)^{-1} f$ to both sides of the last equality, we get

$$K^* f + (L_N^*)^{-1} f = L_N L_N^{-1} (L_N^*)^{-1} f + \widehat{L} K^* L_N^{-1} f + L_N (L_N^*)^{-1} K f + \widehat{L} K^* K f.$$

It follows that

$$(L^*)^{-1} f = L (L^*)^{-1} L^{-1} f \quad \text{for all } f \in H.$$

Thus

$$L^{-1} (L^*)^{-1} f = (L^*)^{-1} L^{-1} f \quad \text{for all } f \in H.$$

The proof is complete. \square

The domain of L_S described as the kernel of the projection $\Gamma_{L_S} = I - L_S^{-1} \widehat{L}$. Here the operator L_S^{-1} takes part in the explicit form. Sometimes there exists another operator T_{L_S} defined on $D(\widehat{L})$ and has the property $\text{Ker } \Gamma_{L_S} = \text{Ker } T_{L_S}$. Between these operators have the following relationship

$$T_{L_S} \Gamma_{L_S} v = T_{L_S} (I - L_S^{-1} \widehat{L}) v = T_{L_S} v - T_{L_S} L_S^{-1} \widehat{L} v = T_{L_S} v \quad \text{for all } v \in D(\widehat{L}).$$

If we know $T_{L_S} v$, then $\Gamma_{L_S} v$ is uniquely determined as the solution of the homogeneous equation $\widehat{L}(\Gamma_{L_S} v) = 0$ with an inhomogeneous condition

$$T_{L_S}(\Gamma_{L_S} v) = T_{L_S} v.$$

Its unique solvability follows from the correctness of the operator L_S . Therefore, it is not necessary to know the explicit form of the operator L_S^{-1} . In the study of differential operators (see [10]) that the operator T_{L_S} is realized in the form of the boundary operator. In such cases we say that the domain is described in terms of the boundary operator. For example, in the case of the Dirichlet problem for a differential equation of elliptic type in $L_2(\Omega)$ that T_{L_S} corresponds to the trace operator on the boundary of Ω , i.e., $T_{L_S} u = u|_{\partial\Omega}$. Therefore it is sufficient to know the form of the boundary operator T_{L_S} . Thus we obtain the following

Corollary 4. *Let there be a correct extension L_S of the minimal operator L_0 with $D(L_S) = D(L_S^*)$, then any other correct extension L has the property $D(L) = D(L^*)$ if and only if $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$ and*

$$T_{L_S} (K^* \widehat{M} - K \widehat{L}) u = 0 \quad \text{for all } u \in D(L), \tag{3.7}$$

where T_{L_S} is a boundary operator corresponding to the fixed correct extension L_S and

$$D(L) = \{u \in D(\widehat{L}) : T_{L_S} (I - K \widehat{L}) u = 0\}.$$

Remark 1. By virtue of the one-to-one mapping of $D(L_S)$ onto $D(L)$:

$$v = (I - K\widehat{L})u \text{ for all } u \in D(L), \quad u = (I + K\widehat{L})v \text{ for all } v \in D(L_S),$$

in practice, sometimes it is more convenient to use the following condition that is equivalent to (3.7):

$$T_{L_S}(K^*\widehat{M} - K\widehat{L} + K^*\widehat{M}K\widehat{L})v = 0 \text{ for all } v \in D(L_S). \quad (3.8)$$

It has the practical convenience because $D(L_S)$ is a fixed domain.

Similarly, we can rephrase Theorem 3 in the following form

Corollary 5. *Let there be one correct normal extension L_N of the formally normal minimal operator L_0 in a Hilbert space H . Then any other correct extension L of L_0 is normal if and only if $L_0 \subset L \subset \widehat{L}$, $R(K) \cup R(K^*) \subset D(\widehat{L}) \cap D(\widehat{M})$,*

$$T_{L_N}(K^*\widehat{M} - K\widehat{L})u = 0 \text{ for all } u \in D(L), \quad (3.9)$$

and

$$\widehat{L}K^* = (\widehat{M}K)^*, \quad (3.10)$$

where T_{L_N} is a boundary operator corresponding to the fixed correct extension L_N and

$$D(L) = \{u \in D(\widehat{L}) : T_{L_N}(I - K\widehat{L})u = 0\},$$

and K is the operator determining the boundary correct extension L from the formula (3.1).

4 The Examples

Example 1. We consider the following operator in a Hilbert space $L_2(0, 1)$

$$\widehat{L}y \equiv y'' + y' = f, \quad (4.1)$$

to which corresponds the minimal operator L_0 with domain

$$D(L_0) = \{y \in W_2^2(0, 1) : y(0) = y(1) = y'(0) = y'(1) = 0\}.$$

We define the operator M_0 as the restriction of \widehat{M} on the set $D(L_0)$. Then the action of the operator M_0 has the form

$$\widehat{M}y \equiv y'' - y' = f.$$

We will denote the maximal operators M_0^* and L_0^* by \widehat{L} and \widehat{M} , respectively. Then we have

$$L_0 \subset \widehat{L}, \quad M_0 \subset \widehat{M} \quad \text{and} \quad D(\widehat{L}) = D(\widehat{M}) = W_2^2(0, 1).$$

Let the operator L_N acts as \widehat{L} with domain

$$D(L_N) = \{y \in D(\widehat{L}) : y(0) + y(1) = 0, \quad y'(0) + y'(1) = 0\}.$$

We take the operator L_N as the fixed correct extensions of L_0 . Note that $D(L_N) = D(L_N^*)$ and $L_0 \subset L_N \subset \widehat{L}$, $M_0 \subset L_N^* \subset \widehat{M}$. The inverse operator to L_N has the form

$$y = L_N^{-1}f = \int_0^x (1 - e^{t-x})f(t)dt - \frac{1}{2} \int_0^1 f(t)dt + \frac{e^{1-x}}{1+e} \int_0^1 e^{t-1}f(t)dt.$$

Then Γ_{L_N} is defined as

$$\Gamma_{L_N} y = \frac{y(0) + y(1)}{2} + \left(\frac{1}{2} - \frac{e^{1-x}}{1+e} \right) [y'(0) + y'(1)].$$

And $\Gamma_{L_N^*} = I - (L_N^*)^{-1} \widehat{M}$ has the following form

$$\Gamma_{L_N^*} y = \frac{y(0) + y(1)}{2} + \left(\frac{e^x}{1+e} - \frac{1}{2} \right) [y'(0) + y'(1)].$$

The correct extension L of L_0 with the property $D(L) = D(L^*)$ is a boundary correct extension. Their inverses are described in the following form

$$y = L^{-1} f = L_N^{-1} f + K f \quad \text{for all } f \in L_2(0, 1),$$

where K is a bounded linear operator in $L_2(0, 1)$ with the properties

$$R(K) \subset \text{Ker } \widehat{L}, \quad R(L_0) \subset \text{Ker } K.$$

In our case, such operators are exhausted by the following operators

$$K f = \int_0^1 f(t) (\overline{a_{11}} + \overline{a_{12}} e^t) dt + e^{-x} \int_0^1 f(t) (\overline{a_{21}} + \overline{a_{22}} e^t) dt,$$

where a_{ij} , $i, j = 1, 2$ are arbitrary complex numbers. Then

$$K^* f = (a_{11} + a_{12} e^x) \int_0^1 f(t) dt + (a_{21} + a_{22} e^x) \int_0^1 e^{-t} f(t) dt.$$

It is known that the direct operator L acts as \widehat{L} from (4.1) and the domain has the form

$$D(L) = \{y \in D(\widehat{L}) : \Gamma_{L_N}(I - K\widehat{L})y = 0\}.$$

In view of Corollary 4, the domain of L can be defined in another way

$$D(L) = \left\{ y \in D(\widehat{L}) : y(0) + y(1) = (K\widehat{L}y)(0) + (K\widehat{L}y)(1), \right. \\ \left. y'(0) + y'(1) = \left(\frac{d}{dx} K\widehat{L}y \right)(0) + \left(\frac{d}{dx} K\widehat{L}y \right)(1) \right\}.$$

First, we will find the correct extensions L such that $D(L) = D(L^*)$. Taking into account Remark 1, let the operator K satisfies the condition (3.8). Then we obtain the system of equations:

$$\left\{ \begin{array}{l} 4(a_{11} + \overline{a_{11}}) + 2(e+1) \left[\frac{\overline{a_{21}}}{e} + a_{12} \right] \cdot A = 0, \\ -4(a_{11} - \overline{a_{11}}) - 2(e+1)(a_{12} - \overline{a_{12}}) - 2 \frac{e+1}{e} (a_{21} - \overline{a_{21}}) \\ - \frac{(e+1)^2}{e} (a_{22} - \overline{a_{22}}) + \left[4a_{12} + 2 \frac{e+1}{e} a_{22} \right] \cdot A = 0, \\ -\frac{1}{e} \overline{a_{21}} + a_{12} + \frac{2}{e} a_{12} \left[\overline{a_{21}}(e-1) + \overline{a_{22}} \frac{e^2-1}{2} \right] = 0, \\ -\frac{1}{e} [2\overline{a_{21}} + \overline{a_{22}}(1+e)] - 2a_{12} - \frac{e+1}{e} a_{22} \\ -4 \frac{a_{12}}{e} \left[\overline{a_{21}} + \overline{a_{22}} \frac{e^2-1}{2} \right] - 2 \frac{e+1}{e^2} a_{22} \left[\overline{a_{21}}(e-1) + \overline{a_{22}} \frac{e^2-1}{2} \right] = 0, \end{array} \right.$$

where

$$A = 2(e - 1)\overline{a_{11}} + (e^2 - 1)\overline{a_{12}} + \frac{e + 1}{e} \left[\overline{a_{21}}(e - 1) + \overline{a_{22}} \frac{e^2 - 1}{2} \right].$$

Solutions of the system of equations with respect to a_{ij} , $i, j = 1, 2$, define the operators K that guarantees the equality $D(L) = D(L^*)$. They will correspond to the following cases:

$$\begin{aligned} I) \quad & D(L) = \left\{ y \in D(\widehat{L}) : y(0) = 0, \quad y(1) = 0 \right\}, \\ II) \quad & D(L) = \left\{ y \in D(\widehat{L}) : y(0) = \frac{a - i}{a + i} y(1), \quad y'(0) = \frac{a - i}{a + i} y'(1), \quad a \in \mathbb{R}, \right. \\ & \quad \left. \text{where } \mathbb{R} \text{ is the space of real numbers} \right\}, \\ III) \quad & D(L) = \left\{ y \in D(\widehat{L}) : ay(0) + \bar{b}y(1) = 0, \quad y(1) = by'(0) + ay'(1), \right. \\ & \quad \left. a \in \mathbb{R}, \quad a \neq 0, \quad b \in \mathbb{C}, \quad |b|^2 = a^2, \quad \text{where } \mathbb{C} \text{ is the space of complex numbers} \right\}. \end{aligned}$$

We use the criterion given in Theorem 3 to find all correct normal extensions L of the minimal operator L_0 . It is easy to verify the formal normality of L_0 and the normality of L_N . The equality $D(L) = D(L^*)$ is necessary for the normality of L . They correspond to three cases of $I) - III)$ described above. Now, if the operator K satisfies (3.3), then the operator L is a normal. The condition (3.3) is equivalent to the following

$$a_{21} = 0, \quad a_{12} = 0.$$

Therefore, the operator K takes the form

$$Kf = \overline{a_{11}} \int_0^1 f(t) dt + \overline{a_{22}} e^{-x} \int_0^1 e^t f(t) dt.$$

Then operators L which act as \widehat{L} from (4.1) turn out to be the normal correct extensions and with the domain

$$D(L) = \left\{ y \in D(\widehat{L}) : y(0) = \frac{a - i}{a + i} y(1), \quad y'(0) = \frac{a - i}{a + i} y'(1), \quad a \in \mathbb{R} \right\}.$$

From three cases of $I) - III)$ are suitable only the case $II)$.

Example 2. Let in the Hilbert space $L_2(\Omega)$, where $\Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, we consider the minimal operator L_0 generated by the Cauchy-Riemann differential operator

$$\widehat{L}u \equiv \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = f(x, y). \quad (4.2)$$

Then

$$D(L_0) = \{u \in W_2^1(\Omega) : T_{L_0}u = 0\},$$

where T_{L_0} is a boundary operator defined as the trace of function $u \in W_2^1(\Omega)$ on the boundary of $\partial\Omega$.

The action of \widehat{M} will have the form

$$\widehat{M}u \equiv -\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = f(x, y).$$

Domains of the operators \widehat{L} and \widehat{M} have the form

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \widehat{L}u \in L_2(\Omega)\},$$

$$D(\widehat{M}) = \{u \in L_2(\Omega) : \widehat{M}u \in L_2(\Omega)\},$$

respectively. If we define the boundary operator T_{L_N} the following way

$$T_{L_N}u = \begin{pmatrix} u(0, y) + u(1, y) \\ u(x, 0) + u(x, 1) \end{pmatrix} \quad \text{for all } u \in D(\widehat{L}),$$

then the operator L_N acting as \widehat{L} with the domain

$$D(L_N) = \{u \in D(\widehat{L}) : T_{L_N}u = 0\},$$

is the correct extension of L_0 . It is easy to verify that L_0 is formally normal and L_N is normal, and in addition $L_0 \subset L \subset \widehat{L}$.

We are interested in the normal boundary correct extensions. Let us clarify some properties of the operator K :

- 1) $R(K) \subset W_2^1(\Omega)$;
- 2) $(Kf)(x + iy)$;
- 3) $(K^*f)(x - iy)$.

The first property follows from the fact that

Assertion 6. *The domain of any normal correct extension L of the minimal operator L_0 generated by the differential operator (4.2) has the property:*

$$D(L) \subset W_2^1(\Omega).$$

Proof. It follows from Theorem 2 of Plesner and Rohlin (see [8]). Now we formulate this theorem: "For each pair of adjoint normal operators A and A^* there exists one and only one pair of self-adjoint operators A_1 and A_2 , satisfying the condition

$$A = A_1 + iA_2, \quad A^* = A_1 - iA_2,$$

where the operators A_1 and A_2 commute". □

The second property follows from the condition $R(K) \subset \text{Ker } \widehat{L}$. The third property follows from the condition $R(L_0) \subset \text{Ker } K$. Further from the conditions (3.9) and (3.10) obtain the operators K for which the correct boundary extension L will be normal.

It follows from Assertion 6 that L_N^{-1} , K , and L^{-1} are compact operators in $L_2(\Omega)$. This means that the normal correct extension L of L_0 is the operator of the discrete spectrum. Hence we have that L has a complete orthonormal system of eigenfunctions.

For clarity, the check of normality by Theorem 3, we consider the special case. Let K will be an integral operator of the form

$$Kf = \int_0^1 \int_0^1 \mathcal{K}(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

It follows from properties 1) and 2) that

$$Kf = \int_0^1 \int_0^1 \mathcal{K}(x + iy, \xi + i\eta) f(\xi, \eta) d\xi d\eta.$$

From the condition (3.3) of Theorem 3, we get that

$$Kf = \int_0^1 \int_0^1 \mathcal{K}(x - \xi + i(y - \eta)) f(\xi, \eta) d\xi d\eta.$$

Using the condition (3.2) of Theorem 3 for the operator K , we obtain all normal correct extensions. We will not give this condition on the kernel $\mathcal{K}(x - \xi + i(y - \eta))$, because of the cumbersome to write.

To demonstrate the mechanism of checking the condition (3.2), we consider the special case when

$$\mathcal{K}(x - \xi + i(y - \eta)) = ae^{i\pi(x - \xi + i(y - \eta))},$$

where $a \in \mathbb{C}$ is a complex number of the form $a = a_1 + ia_2$. Then the condition (3.2) is equivalent to

$$2a_2 + (a_1^2 + a_2^2)(e^\pi - e^{-\pi}) = 0.$$

There are two kinds of solutions of this equation:

$$\begin{aligned} I. \quad & a_1 = 0, \quad a_2 = \frac{2}{e^{-\pi} - e^\pi}; \\ II. \quad & a_2 = \frac{-1 \pm \sqrt{1 - [a_1(e^\pi - e^{-\pi})]^2}}{e^\pi - e^{-\pi}}, \quad \text{where} \quad |a_1| \leq \frac{1}{e^\pi - e^{-\pi}}. \end{aligned}$$

Then in the case of II , the correct extension corresponding to the following boundary problem

$$\begin{aligned} \widehat{L}u &\equiv \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = f(x, y) \quad \text{for all } f \in L_2(\Omega), \\ D(L) &= \left\{ u \in W_2^1(\Omega) : u(0, y) + u(1, y) = 0, \quad 0 \leq y \leq 1, \right. \\ &\quad \left. u(x, 0) + u(x, 1) = ia(e^\pi + 1) \int_0^1 e^{i\pi(x - \xi)} u(\xi, 1) d\xi \right. \\ &\quad \left. - ia(e^{-\pi} + 1) \int_0^1 e^{i\pi(x - \xi)} u(\xi, 0) d\xi, \quad 0 \leq x \leq 1 \right\} \end{aligned}$$

is normal, where $a = a_1 + ia_2$, or in the case of I , the correct extension corresponding to the boundary problem

$$\begin{aligned} D(L) &= \left\{ u \in W_2^1(\Omega) : u(0, y) + u(1, y) = 0, \right. \\ &\quad \left. u(x, 0) + u(x, 1) = 2 \int_0^1 e^{i\pi(x - \xi)} u(\xi, 1) d\xi \right\}, \end{aligned}$$

is normal.

All normal correct extensions L have a compact inverse operator because of $D(L) \subset W_2^1(\Omega)$. Therefore, their eigenfunctions create an orthonormal basis in $L_2(\Omega)$. In the particular case when

$$\mathcal{K}(x, y; \xi, \eta) = \frac{2i}{e^{-\pi} - e^\pi} \cdot e^{i\pi(x - \xi + i(y - \eta))},$$

we obtain the orthonormal basis in the following form:

$$u_{k,n}(x, y) = \begin{cases} e^{2n\pi iy + i\pi x}, & n = 0, \pm 1, \pm 2, \dots \\ e^{(2k+1)\pi ix + (2n+1)\pi iy}, & k = \pm 1, \pm 2, \dots, n = 0, \pm 1, \pm 2, \dots \end{cases}$$

and the corresponding eigenvalues

$$\lambda_{k,n} = \begin{cases} i\pi - 2n\pi, & n = 0, \pm 1, \pm 2, \dots \\ (2k+1)\pi i - (2n+1)\pi, & k = \pm 1, \pm 2, \dots, n = 0, \pm 1, \pm 2, \dots \end{cases}$$

Thus, this method allows us to check for normality of an unbounded operator. Preliminary it is necessary to clarify the question of the existence of at least one normal extension. For the existence of a normal extension we need that the minimal operator must be formally normal.

Remark 2. If in Example 2 the square area Ω is replaced by the unit circle, then the minimal operator L_0 will not be formally normal. Thus in this case, there are no normal extensions of L_0 in $L_2(\Omega)$.

Remark 3. When the minimal operator L_0 is symmetric and the fixed operator L_N is self-adjoint then the conditions of Theorem 3 are equivalent to $K = K^*$ and we have all the self-adjoint correct extensions.

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